Measures of Fractal Dimensions

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Abstract

In this paper, we will introduce the reader to the Hausdorff dimension, the packing dimension, and the Mandelbrot dimension. All three of these can be used to measure fractals and less exotic sets as well. We will show computations of these dimensions using examples such as the Koch Curve and the set of rational numbers.

1 Introduction

The notion of dimension is surprisingly tricky. It is quite easy to tell the dimension of some of the more familiar geometric shapes and sets–points have dimension zero, sinusoidal curves have dimension one, planes have dimension two, et cetera. Intuition suggests that an n-dimensional object should be described by n parameters. However, this idea fails easily.

Example 1. *We shall here describe $\mathbb{C}$ as the image of a function of $\mathbb{R}$. Let f map $\mathbb{R}$ to $\mathbb{C}$. For each $x \in \mathbb{R}$, let $\{a_k(x)\}_{k=-\infty}^{\infty}$ such that the following conditions hold:*

$$\forall k \in \mathbb{Z},\ a_k(x) \in \{0, 1\}$$

$$\forall n \in \mathbb{Z}\ \exists m < n,\ a_m(x) \neq 0$$

$$x = \sum_{k=-\infty}^{\infty} a_k(x) 2^k$$

This simply constructs a unique binary expansion for the real number x. We now set our function:

$$f(x) = \sum_{k=-\infty}^{\infty} a_{2k}(x) 2^k + i a_{2k+1}(x) 2^k$$

Though f is a function of a single variable, its image is clearly two-dimensional.

Furthermore, there are more exotic sets demanding consideration. For example, the Koch Curve has an infinite length in a finite area. Sets like this are counter-intuitive by nature, so we can not rely on intuition to determine the properties of these sets.

Before proceeding to an introduction to some of the popular methods used to evaluate the dimension of fractals, it is important to note that these methods are not just for artificial paradoxical cases. Quoth Benoit Mandelbrot, "Mountains are not cones, clouds are not spheres... nor does lightning travel in a straight line." Nature in many ways does mimic fractal patterns, for example in the shape of cauliflower or the surface of an electrode. So, an understanding of fractals is requisite for an understanding of nature.[2]

2 Hausdorff Dimension

Since many fractals, such as Julia Sets and the Koch Curve, exhibit self-similarity, and the dimension of a fractal can be considered as a measure of complexity, a notion of dimension based on self-similarity may be useful. However, this cannot measure other types of fractals such as Brownian trajectories, strange attractors, or many fractal surfaces found in nature. We shall here show how this dimension of internal similarity can be calculated and used.

Suppose $\{f_j\}_{j=1}^m$ is a set of contracting similarity maps on $\mathbb{R}^n$. From the definition of a contracting map, we have for all $i \in \mathbb{N}_{\leq m}$ there exists some $r_i \in (0,1)$ such that for all $x, y \in \mathbb{R}^n$:

$$|f_i(x) - f_i(y)| = r_i|x-y|$$

Furthermore, let there exist a nonempty open set V such that $\bigsqcup_{i=1}^{m} f_i(V) \subseteq V$. This is called the open set condition. [1] Note that $\bigsqcup$ is the symbol for disjoint union.

Let X be a fractal set such that $X = \bigcup_{i=1}^{m} f_i(X)$. The dimension of X is the solution d to $\sum_{k=1}^{m} r_k^d$. Recall that r_i is the contraction ratio of f_i.[3]

Example 2. *The familiar Cantor Ternary Set, the construction of which is shown below, has a non-integer dimension which can be easily calculated by this method.*

We set $f_1(x) = \frac{x}{3}$ and $f_2(x) = \frac{2+x}{3}$. It is simple to check that f_1 and f_2 are similarity maps for the Cantor Ternary Set C.

$$C = f_1(C) \cup f_2(C)$$

We can also easily check that f_1 and f_2 are contractive:

$$\begin{aligned}|f_1(x) - f_1(y)| &= \left|\frac{x}{3} - \frac{y}{3}\right| \\ &= \frac{1}{3}|x-y|\end{aligned}$$

$$
\begin{aligned}
|f_2(x) - f_2(y)| =& \left|\frac{x+2}{3} - \frac{y+2}{3}\right| \\
=& \left|\frac{x}{3} + \frac{2}{3} - \frac{y}{3} - \frac{2}{3}\right| \\
=& \left|\frac{x}{3} - \frac{y}{3}\right| \\
=& \frac{1}{3}|x-y|
\end{aligned}
$$

It is easy to see that $r_1 = r_2 = \frac{1}{3}$. *The open set condition is likewise easy to check: we simply try* $(0,1)$ *as the feasible open set.*

$$
\begin{aligned}
f_1((0,1)) \cup f_2((0,1)) &= \left(0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right) \\
&\subset (0,1)
\end{aligned}
$$

$$
\begin{aligned}
f_1((0,1)) \cap f_2((0,1)) =& \left(0, \frac{1}{3}\right) \cap \left(\frac{2}{3}, 1\right) \\
=& \emptyset
\end{aligned}
$$

We have thus verified that this method of finding the dimension applies. We can now simply solve our equation:

$$
\begin{aligned}
1 =& r_1^d + r_2^d \\
=& 2\frac{1}{3^d} \\
3^d &= 2 \\
d \ln 3 &= \ln 2 \\
d =& \frac{\ln 2}{\ln 3} \\
\approx& 0.631
\end{aligned}
$$

This result does make sense given our knowledge of the Cantor Ternary Set. The set is well known to be uncountable in cardinality, so it should have a dimension greater than that of a point. However, the set is a collection of isolated points, so it should have a dimension lower than that of a line. Thus, this notion of dimension logically characterizes the rather enigmatic Cantor Set.

As it turns out, this dimension of internal similarity is equivalent to what is known as the Hausdorff dimension where it applies[3]. This can be thought of as a sphere-counting notion of dimension, and its definition is somewhat complicated. However, being based on a notion of distance, the Hausdorff dimension has the advantage of applying to any subset of a metric space, Euclidean or not.

We here show how to compute the Hausdorff dimension, henceforth denoted by d_H.

Let X be a metric space, d be the distance function on X, and $A \subset X$. It is here necessary to define the diameter of a subset of X:

$$\delta(B) = \sup\{d(x, y) : x \in B, y \in B\}$$

For a given $p \in \mathbb{R}$ and $\epsilon > 0$, we set:

$$m_p^\epsilon(A) = \inf\{\sum_{k=1}^{\infty}(\delta(A_k))^p : A = \bigsqcup_{k=1}^{\infty} A_k, \delta(A_k) < \epsilon\}$$

We can now use this definition to formulate the Hausdorff dimension.[2]

$$m_p(A) = \sup\{m_p^\epsilon(A) : \epsilon > 0\}$$

$$d_H = \inf\{p \in \mathbb{R} : m_p(A) = 0\}$$

This may seem a bit complicated, but the computation is much more clear in the context of an example.

Example 3. *We here compute the Hausdorff dimension of* $A = [0, 1] \times [0, 1]$*. For the partitioning, we use a grid of very small squares for values of* p *greater than two. We partition* A *into* $\frac{\epsilon}{\sqrt{2}} \times \frac{\epsilon}{\sqrt{2}}$*-sized squares for values of* $p < 2$ *and* $\frac{\epsilon}{\sqrt{2}} \in \{\frac{1}{n} : n \in \mathbb{N}\}$*. For these values of epsilon, we have:*

$$m_p^\epsilon(A) = \begin{cases} 0 & : p > 2 \\ 1 & : p = 2 \\ \epsilon^p \times \frac{\sqrt{2}}{\epsilon} \times \frac{\sqrt{2}}{\epsilon} & : p < 2 \end{cases}$$

$$m_p(A) = \begin{cases} 0 & : p > 2 \\ 1 & : p = 2 \\ \infty & : p < 2 \end{cases}$$

Since $m_p(A) = 0$ *only for* $p \in (2, \infty)$*,* $d_H = 2$*. This is exactly what we would expect for the dimension of a planar area. Note that the expression for* $m_p^\epsilon(A)$ *may be inaccurate for small values of* p*, since we did not consider every possible partition of* A*. However, the result still holds because the true value of* $m_p^\epsilon(A)$ *is surely nonzero for* $p < 2$*.*

3 Mandelbrot Dimension

The Hausdorff dimension is not the only popular notion of dimension. What is called the Mandelbrot dimension, also known as capacity, Schnirelman-Kolmogorov dimension, upper metric dimension, or the fractal dimension, is somewhat easier to grasp geometrically than the Hausdorff dimension. Like the Hausdorff dimension, the Mandelbrot dimension is based on the cardinality of covering sets and can be applied to any subset of a metric space. We here denote the Mandelbrot dimension by d_M.

We first define our covering-counting function as the smallest number of sets with diameter no more than ϵ needed to cover A:

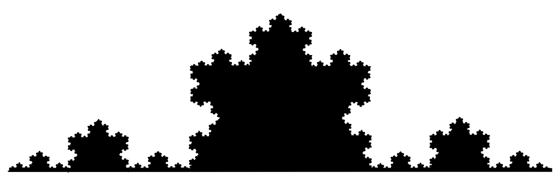
$$N_A(\epsilon) = \inf\{|\{A_k\}| : A \subseteq \bigcup_k A_k, \delta(A_k) \leq \epsilon\}$$

We can now use this to compute the Mandelbrot dimension of A[2]:

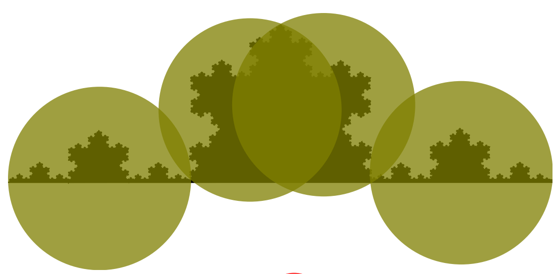
$$d_M = \limsup_{\epsilon \to 0} \frac{-\ln N_A(\epsilon)}{\ln \epsilon}$$

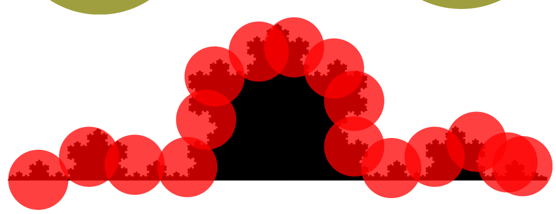
The Mandelbrot dimension often coincides with the Hausdorff dimension, but this is not always the case. As it turns out, the Mandelbrot dimension is always greater than or equal to the Hausdorff dimension.[4]

Example 4. *We here demonstrate the computation of the Mandelbrot dimension in the context of the Koch Curve, κ, which is the boundary of the object shown below. We take the endpoints of κ to be $(0,0)$ and $(1,0)$.*



It is easy to see that if $\epsilon = \frac{1}{3^n}$, then $N_\kappa(\epsilon) = 4^n$.





We can now compute the dimension directly:

$$\begin{aligned} d_M &= \limsup_{\epsilon\to 0} \frac{-\ln N_\kappa(\epsilon)}{\ln \epsilon} \\ &= \lim_{n\to\infty} \frac{-\ln 4^n}{\ln 3^{-n}} \\ &= \frac{\ln 4}{\ln 3} \\ &\approx 1.262 \end{aligned}$$

4 Packing Dimension

The packing dimension is another general notion of the dimension of a set in a metric space. This is based on packing spheres into the measured set. The other steps in the computation are reminiscent of the calculation of the Hausdorff dimension, which may make it difficult to understand and use. However, it is unique and somewhat popular, and thus is certainly worth understanding. We shall denote the packing dimension by d_P.

We use the following definitions to compute the packing dimension[5][2]:

$$\Lambda_{\alpha,\epsilon}(A) = \sup\{\sum_j (\delta(B_j))^\alpha : \{B_j\} \text{ is a disjoint set of spheres with centers in } A, \delta(B_j) < 2\epsilon\}$$

$$\Lambda_\alpha(A) = \lim_{\epsilon\to 0} \Lambda_{\alpha,\epsilon}(A)$$

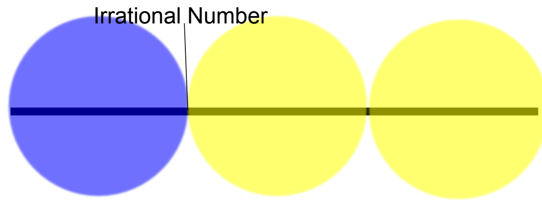
$$\Lambda'_\alpha(A) = \inf\{\sum_{n=1}^{\infty} \Lambda_\alpha(A_n) : A \subseteq \bigcup_{n=1}^{\infty} A_n\}$$

$$\begin{aligned} d_P &= \inf\{\alpha > 0 : \Lambda'_\alpha(A) = 0\} \\ &= \sup\{\alpha > 0 : \Lambda'_\alpha(A) = \infty\} \end{aligned}$$

This is a different notion of dimension with its own properties that are still being investigated. In relation to the other notions of dimension, we have[2]:

$$d_H \leq d_P \leq d_M$$

Example 5. *We shall here demonstrate the computation of the packing dimension by finding the dimension of $\mathbb{Q} \cap [0,1]$, which we here name R. We know that both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are everywhere dense, so we can choose our open spheres in such a way that their boundary points coincide only at rational numbers. Because these points are everywhere, we can essentially ignore said criterion when calculating the pre-measure. In this calculation, we only consider cases where $\frac{1}{2\epsilon} \in \mathbb{N}$.*



$$\Lambda_{\alpha,\epsilon}(R) = \begin{cases} (2\epsilon)^{\alpha-1} & : \alpha > 1 \\ 1 & : \alpha = 1 \\ \infty & : \alpha < 1 \end{cases}$$

$$\Lambda_{\alpha}(R) = \begin{cases} 0 & : \alpha > 1 \\ 1 & : \alpha = 1 \\ \infty & : \alpha < 1 \end{cases}$$

$$d_p = 1$$

Here is an instance where the packing dimension and the Hausdorff dimension disagree. It is easy to check that the Hausdorff dimension of R is zero, but we have shown that the packing dimension is one.

5 Conclusion

We have presented and demonstrated three notions of dimension: the Hausdorff dimension, the Mandelbrot dimension, and the packing dimension. These three concepts may seem very similar, but it has been shown that they may yield different results. This leaves the question of dimension open–which of these measurements is the best? Regardless of whichever may be the most useful in a specific application, we can now measure even fractal sets without relying on intuition.

References

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